FIRST-ORDER LOGIC

Lara J. Martin (she/they)
TA: Aydin Ayanzadeh (he)

10/5/2023

CMSC 671

By the end of class today, you will be able to:

- Convert natural language sentences into first-order logic

Modified from slides by Dr. Cynthia Matuszek

LOGISTICS

Redownload the file for HW 2. There are some clarifications:
https://laramartin.net/Principles-of-AI/homeworks/logic/logical-agent.html

HW 1 should be graded by tomorrow

RECAP

PROPOSITIONAL LOGIC

- Semantics:
  - Facts about the world that are true or false
- Syntax:
  - Propositional symbols (stand for statements)
    - E.g., W13, FacingEast, IsCloudyToday
  - Logical Connectives (turning simple sentences into complex sentences)
    - E.g., $\neg$, $\land$,$\lor$,$\Rightarrow$,$\Leftrightarrow$

FIRST-ORDER LOGIC

FIRST-ORDER LOGIC (FOL)

- First-order logic (like natural language) assumes the world contains
  - **Objects:** people, houses, numbers, colors, baseball games, wars, …
  - **Functions:** father of, best friend, one more than, plus, …
    - Function arguments are objects; function returns an object
  - Objects generally correspond to English NOUNS

  - **Predicates/Relations/Properties:** red, round, prime, brother of, bigger than, part of, between…
    - Predicate arguments are objects; predicate returns a truth value
  - Predicates generally correspond to English VERBS
    - **First argument is generally the subject, the second the object**
    - Hit(Bill, Ball) usually means "Bill hit the ball."
    - Likes(Bill, IceCream) usually means "Bill likes IceCream."
    - Verb(Noun1, Noun2) usually means "Noun1 verb noun2."

OR RATHER…

- **Objects:** things with individual identities
- **Properties:** attributes of objects that distinguish them from other objects
- **Relations:** hold among sets of objects
- **Functions:** subset of relations where there is only one “value” for any given “input”

SYNTAX OF FOL: BASIC ELEMENTS

• Constants	KingJohn, 2, UMBC,...	
• Predicates	BrotherOf, >,...	(return true or false)
• Functions	Sqrt, LeftLegOf,...	(return some object)
• Variables	x, y, a, b,...	
• Quantifiers	$\forall$, $\exists$	
• Connectives	$\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$	(standard)
• Equality	$=$	(but causes difficulties….)

SYNTAX USER PROVIDES…

- **Constant symbols:** represent individuals in the world
  - Mary
  - 3
  - Green
- **Function symbols:** map individuals to individuals
  - father-of(Mary) = John
  - color-of(Sky) = Blue
- **Predicate symbols:** map individuals to truth values
  - greater(5,3)
  - green(Grass)
  - color(Grass, Green)

SYNTAX FOL PROVIDES…

- **Variable symbols**
  - E.g., x, y, foo
- **Connectives**
  - Same as in PL: not (¬), and (∧), or (∨), implies (⇒), if and only if (biconditional ↔)
- **Quantifiers**
  - Universal ∀x or (Ax)
  - Existential ∃x or (Ex)

FOL SENTENCES

- A **term** (denoting a real-world individual) is:
  - A constant symbol: *John*, or
  - A variable symbol: *x*, or
  - An n-place **function** of n terms

    x and $f(x_1, ..., x_n)$ are terms, where each x_i is a term

    is-a(John, Farmer)
  - A term with no variables is a **ground term**.
- An **atomic sentence** is an n-place **predicate** of n terms
  - Has a truth value (*t* or *f*)

Like it's the "base"

FOL VS PROPOSITIONAL LOGIC

Propositional Logic

- Assumes world contains facts
- Syntax: sentences & connectives

First-Order Logic

- Assumes world contains facts, objects, relations
- Syntax: variables, quantifiers, user-defined structures

SYNTAX OF FOL: ATOMIC SENTENCES

- Atomic sentences in logic state facts that are true or false.
- Properties and m-ary relations do just that:
  - LargerThan(2, 3) is false.
  - BrotherOf(Mary, Jane) is false.
  - Married(Father(Richard), Mother(John)) could be true or false.
- Note: Functions refer to objects, do not state facts, and form no sentence:
  - Brother(Pete) refers to John (his brother) and is neither true nor false.
  - Plus(2, 3) refers to the number 5 and is neither true nor false.
- BrotherOf(Pete, Brother(Pete)) is True.

Binary relation is a truth value.

Function refers to John, an object in the world, i.e., John is Pete's brother.

SYNTAX OF FOL: VARIABLES

- **Variables** range over objects in the world.
- A **variable** is like a **term** because it represents an object.
- A **variable** may be used wherever a **term** may be used.
  - **Variables** may be arguments to functions and predicates.

(A **term with NO variables** is called a **ground term**.)
(A **variable not bound by a quantifier** is called **free**.)

SYNTAX OF FOL: BASIC SYNTAX ELEMENTS ARE SYMBOLS

- Constant Symbols (correspond to English nouns)
  - Stand for objects in the world. E.g., KingJohn, 2, France, ...
- Predicate Symbols (correspond to English verbs)
  - Stand for relations (maps a tuple of objects to a truth-value)
    - E.g., Brother(Richard, John), greater_than(3,2), ...
- Function Symbols (correspond to English nouns)
  - Stand for functions (maps a tuple of objects to an object)
    - E.g., Sqrt(3), LeftLegOf(John), ...
- Model (world) = set of domain objects, relations, functions
- Interpretation maps symbols onto the model (world)
  - Very many interpretations are possible for each KB and world!
  - Job of the KB is to rule out models inconsistent with our knowledge.

SENTENCES: TERMS AND ATOMS

- A complex sentence is formed from atomic sentences connected by the same logical connectives as in propositional logic:
  $\neg P$, $P \vee Q$, $P \wedge Q$, $P \Rightarrow Q$, $P \leftrightarrow Q$ where P and Q are sentences
- *has-a(x, Bachelors)* ∧ *is-a(x, human)*

- *has-a(John, Bachelors)* ∧ *is-a(John, human)*
- *has-a(Mary, Bachelors)*

does NOT SAY everyone with a bachelors' is human

QUANTIFIERS

- **Universal quantification**
  - ∀x P(x) means that P holds for **all** values of *x* in its domain
  - States universal truths
  - E.g.: $\forall x\ dolphin(x) \Rightarrow mammal(x)$
- **Existential quantification**
  - ∃x P(x) means that P holds for **some** value of x in the domain associated with that variable
  - Makes a statement about some object without naming it
  - E.g., $\exists x\ mammal(x) \land lays\text{-}eggs(x)$

SENTENCES: QUANTIFICATION

- **Quantified sentences** adds quantifiers $\forall$ and $\exists$
- $\forall x\ \textit{has-a}(x, \textit{Bachelors}) \Rightarrow \textit{is-a}(x, \textit{human})$
- $\exists x\ \textit{has-a}(x, \textit{Bachelors})$
- $\forall x\ \exists y\ \textit{Loves}(x, y)$

Everyone who has a bachelors' is human.

There exists some who has a bachelors'.

Everybody loves somebody.

QUANTIFIERS: USES OF ∀

- Universal quantifiers **often** used with "implies" to form "rules":

  $(\forall x)\ student(x) \Rightarrow smart(x)$

  "All students are smart"

- Universal quantification **rarely*** used to make blanket statements about every individual in the world:

  $(\forall x) student(x) \wedge smart(x)$

  "Everyone in the world is a student and is smart"

*Deliberately, anyway

QUANTIFIERS: USES OF ∃

- Existential quantifiers are **usually** used with "and" to specify a list of properties about an individual:
  - $(\exists x)\ student(x) \wedge smart(x)$
  - "There is a student who is smart"
- A common mistake is to represent this English sentence as the FOL sentence:
  - $(\exists x)\ student(x) \Rightarrow smart(x)$
  - But what happens when there is a person who is not a student?

TRANSLATION WITH QUANTIFIERS

- Universal statements typically use **implications**
  - All S(x) is P(x):
    - $\forall x(S(x) \Rightarrow P(x))$
  - No S(x) is P(x):
    - $\forall x(S(x) \Rightarrow \neg P(x))$
- Existential statements typically use **conjunctions**
  - Some S(x) is P(x):
    - $\exists x (S(x) \wedge P(x))$
  - Some S(x) is not P(x):
    - $\exists x (S(x) \wedge \neg P(x))$

SENTENCES: WELL-FORMEDNESS

- A **well-formed formula (wff)** is a sentence containing no "free" variables. That is, all variables are "bound" by universal or existential quantifiers.

- $(\forall x)P(x,y)$ has x bound as a universally quantified variable, but y is free.

QUANTIFIER SCOPE

- Switching the order of universal quantifiers does not change the meaning:
  - $(\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x)\ P(x,y)$
- Similarly, you can switch the order of existential quantifiers:
  - $(\exists x)(\exists y)P(x,y) \leftrightarrow (\exists y)(\exists x)\ P(x,y)$
- Switching the order of universals and existentials does change meaning:
  - Everyone likes someone: $(\forall x)(\exists y)$ likes(x,y)
  - Someone is liked by everyone: $(\exists y)(\forall x)$ likes(x,y)

CONNECTIONS BETWEEN ALL AND EXISTS

- We can relate sentences involving ∀ and ∃ using De Morgan's laws:
  - $(\forall x)\, \neg P(x) \leftrightarrow \neg(\exists x)\, P(x)$
  - $\neg(\forall x)\, P \leftrightarrow (\exists x)\, \neg P(x)$
  - $(\forall x)\, P(x) \leftrightarrow \neg(\exists x)\, \neg P(x)$
  - $(\exists x)\, P(x) \leftrightarrow \neg(\forall x)\, \neg P(x)$

QUANTIFIED INFERENCE RULES

- Universal instantiation
  - $\forall x\ P(x)\ \therefore\ P(A)$
- Universal generalization
  - $P(A) \land P(B) \ldots \therefore\ \forall x\ P(x)$
- Existential instantiation
  - $\exists x\ P(x)\ \therefore P(F)$ **← skolem constant F**
- Existential generalization
  - $P(A)\ \therefore\ \exists x\ P(x)$

TRANSLATING ENGLISH TO FOL

- **Every gardener likes the sun.**
  - $\forall x\ \text{gardener}(x) \Rightarrow \text{likes}(x,\text{Sun})$
- **You can fool some of the people all of the time.**
  - $\exists x\ \forall t\ \ \text{person}(x) \wedge \text{time}(t) \Rightarrow \text{can-fool}(x,t)$
- **You can fool all of the people some of the time.**
  - $\forall x\ \exists t\ (\text{person}(x) \Rightarrow \text{time}(t) \wedge \text{can-fool}(x,t))$
  - $\forall x\ (\text{person}(x) \Rightarrow \exists t\ (\text{time}(t) \wedge \text{can-fool}(x,t))$

  Equivalent (the two formulas above)
- **All purple mushrooms are poisonous.**
  - $\forall x\ (\text{mushroom}(x) \wedge \text{purple}(x)) \Rightarrow \text{poisonous}(x)$

TRANSLATING ENGLISH TO FOL

- **No purple mushroom is poisonous.**
  - $\neg\exists x$ purple(x) $\wedge$ mushroom(x) $\wedge$ poisonous(x)
  - $\forall x$ (mushroom(x) $\wedge$ purple(x)) $\Rightarrow$ $\neg$poisonous(x)

  Equivalent
- **There are exactly two purple mushrooms.**
  - $\exists x\ \exists y$ mushroom(x) $\wedge$ purple(x) $\wedge$ mushroom(y) $\wedge$ purple(y) ^ $\neg$(x=y) $\wedge$ $\forall z$ (mushroom(z) $\wedge$ purple(z)) $\Rightarrow$ ((x=z) $\vee$ (y=z))
- **Mary is not tall.**
  - $\neg$tall(Mary)
- **X is above Y iff X is on directly on top of Y or there is a pile of one or more other objects directly on top of one another starting with X and ending with Y.**
  - $\forall x\ \forall y$ above(x,y) $\leftrightarrow$ (on(x,y) $\vee$ $\exists z$ (on(x,z) $\wedge$ above(z,y)))

NECESSARY AND SUFFICIENT

- p is **necessary** for q
  - $\neg p \Rightarrow \neg q$ (“no p, no q!”)
- p is **sufficient** for q
  - $p \Rightarrow q$ (“p is all we need to know!”)
- Note that $\neg p \Rightarrow \neg q$ is equivalent to $q \Rightarrow p$
- So if p is necessary and sufficient for q, then p iff q.

MORE ON DEFINITIONS

- Examples: define father(x, y) by parent(x, y) and male(x)
  - parent(x, y) is a **necessary** (**but not sufficient**) description of father(x, y)
    - father(x, y) ⇒ parent(x, y)
  - parent(x, y) ^ male(x) ^ age(x, 35) is a **sufficient** (**but not necessary**) description of father(x, y):
    - father(x, y) ← parent(x, y) ^ male(x) ^ age(x, 35)
  - parent(x, y) ^ male(x) is a **necessary and sufficient** description of father(x, y)
    - parent(x, y) ^ male(x) ↔ father(x, y)

EXERCISE: FOL TRANSLATION

1. Everything is bitter or sweet.
2. Either everything is bitter or everything is sweet.
3. There is somebody who is loved by everyone.
4. Nobody is loved by no one.
5. If someone is noisy, everybody is annoyed
6. Frogs are green.
7. Frogs are not green.
8. No frog is green.
9. Some frogs are not green.
10. A mechanic likes Bob.
11. A mechanic likes herself.
12. Every mechanic likes Bob.
13. Some mechanic likes every nurse.
14. There is a mechanic who is liked by every nurse.

Exercises: disi.unitn.it/~bernardi/Courses/LSNL/Slides/fl1.pdf

EXERCISE: FOL TRANSLATION

1. Everything is bitter or sweet.
2. Either everything is bitter or everything is sweet.
3. There is somebody who is loved by everyone.
4. Nobody is loved by no one.
5. If someone is noisy, everybody is annoyed
6. Frogs are green.
7. Frogs are not green.

1. $\forall x\ (bitter(x) \lor sweet(x))$
2. $\forall x\ (bitter(x)) \lor \forall x\ (sweet(x))$
3. $\exists x\ \forall y\ (loves(y,x))$
4. $\neg\exists x\ \neg\exists y\ (loves(y,x))$
5. $\exists x\ (noisy(x)) \Rightarrow \forall y(annoyed(y))$
6. $\forall x\ (frog(x) \Rightarrow green(x))$
7. $\forall x\ (frog(x) \Rightarrow \neg green(x))$

Exercises: disi.unitn.it/~bernardi/Courses/LSNL/Slides/fl1.pdf

EXERCISE: FOL TRANSLATION

8. No frog is green.
9. Some frogs are not green.
10. A mechanic likes Bob.
11. A mechanic likes herself.
12. Every mechanic likes Bob.
13. Some mechanic likes every nurse.
14. There is a mechanic who is liked by every nurse.

8. $\neg\exists x\, (frog(x) \wedge green(x))$
9. $\exists x\, (frog(x) \wedge \neg green(x))$
10. $\exists x\, (mech.(x) \wedge likes(x, Bob))$
11. $\exists x\, (mech.(x) \wedge likes(x, x))$
12. $\forall x\, (mech.(x) \Rightarrow likes(x, Bob))$
13. $\exists x\, \forall y\, (mech(x) \wedge nurse(y) \Rightarrow likes(x, y))$
14. $\exists x(mech(x) \wedge \forall y(nurse(y) \Rightarrow likes(y, x))$

Exercises: disi.unitn.it/~bernardi/Courses/LSNL/Slides/fl1.pdf

FOR NEXT CLASS

- Read Chapters 11.1-11.7
- Work on Homework 2